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On R^4 threshold corrections in type IIB string theory and (p,q)-string instantons

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ABSTRACT

We obtain the exact non-perturbative thresholds of R^4 terms in type IIB string theory compactified to eight and seven dimensions. These thresholds are given by the perturbative tree-level and one-loop results together with the contribution of the D-instantons and of the (p,q)-string instantons. The invariance under U-duality is made manifest by rewriting the sum as a non-holomorphic invariant modular function of the corresponding discrete U-duality group. In the eight-dimensional case, the threshold is the sum of a order-1 Eisenstein series for $SL(2,\mathbb{Z})$ and a order-3/2 Eisenstein series for $SL(3,\mathbb{Z})$. The seven-dimensional result is given by the order-3/2 Eisenstein series for $SL(5,\mathbb{Z})$. We also conjecture formulae for the non-perturbative thresholds in lower dimensional compactifications and discuss the relation with M-theory.

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1 Introduction

Important progress has recently been made towards understanding the non-perturbative structure of string theory. Extended supersymmetry implies non-perturbative equivalence between string theories. Quantitative tests, although scarce, can be carried out by computing threshold corrections to specials terms in the effective action, which are "BPS saturated". Such terms receive perturbative corrections from (at most) a single order in perturbation theory, to which only BPS states contribute. Usually they are related by supersymmetry to anomaly cancelling terms. Moreover, they receive instanton corrections from special instanton configurations that leave some of the supersymmetries unbroken. Examples are D-terms in N=1 theories in four dimensions, the two derivative terms in N=1 six-dimensional theories, the F^4 and R^4 terms in theories with N=1 ten-dimensional supersymmetry [1, 2] and R^4 terms in ten-dimensional N=2 theories [3].

Thresholds of "BPS saturated" terms become more complicated as the theory is compactified to lower dimension without breaking the supersymmetry. The reason is that the number of scalar moduli that they can depend upon grows, and the duality symmetries become larger as one decreases the number of non-compact dimensions.

Here we will mainly focus on the various R^4 terms in type IIB string theory toroidally compactified to eight and seven dimensions. In ten dimensions such terms have tree-level and one-loop corrections that have been computed [3] and are believed to receive no other perturbative corrections. On the other hand, the type IIB string has instantons already in ten dimensions, known as D-instantons. Their contribution was conjectured in Ref. [3] on the basis of the $SL(2,\mathbb{Z})$ symmetry of the type IIB theory. Upon compactification on a circle, nothing exotic happens. Since the D-instanton contributions are independent of the non-compact dimensions the nine-dimensional thresholds can be obtained from the ten-dimensional exact result and from the nine-dimensional perturbation theory [3].

Upon compactification to eight dimensions a new type of instanton enters the game, namely (Euclidean) (p,q)-strings whose world-sheets wrap around the target space torus. One of the main points in this paper is to calculate their contributions from first principles and show that the full result is $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$ invariant (the U-duality group in eight dimensions). This result also matches a recent proposal for the same threshold calculated in the context of M-theory [4].

Compactifying type II string theory further down to seven dimensions, we do not expect anything exotic to happen. The only difference is that now the Euclidean (p,q) instantons are wrapping in all possible ways on two dimensional submanifolds of the three-torus. This contribution can be evaluated from the perturbative world-sheet instantons of the fundamental type IIB string on T^3. This will enable us to derive an expression for the exact R^4 threshold in seven dimensions, that exhibits manifest $SL(5,\mathbb{Z})$ U-duality symmetry.

Generalizing the above pattern, we will also propose an exact expression for the six-dimensional case. This expression is manifestly invariant under the $SO(5,5,\mathbb{Z})$ U-duality group, and should reproduce the contributions of the D-instantons and D(p,q)-strings, together with the contribution of the D-3brane that can be wrapped on the four-torus. Lower dimensional cases lead us into the realm of discrete exceptional groups, for which we will have little to say here.

Our proposal for the threshold is as follows. Let G/H be the homogeneous space describing the scalars of a given compactification. When this coset space is irreducible, the kinetic terms of the scalars can be written in terms of the matrix M in the adjoint representation of G as

$$S_{\text{scalars}} = \int d^D x \; tr(\partial M \partial M^{-1}) \tag{1.1}$$

Then, we conjecture that the threshold will be given by the order-3/2 Eisenstein series

$$E_{\frac{3}{2}}(M) = \sum_{m^i} \left[\sum_{i,j} m^i M_{ij} m^j \right]^{-\frac{3}{2}} \tag{1.2}$$

The cases $D = 9$ and $D = 8$ have the peculiarity that the coset space is reducible. The nine-dimensional case was investigated in Ref. [17]. In eight dimensions, which we analyze in detail here, the scalar manifold splits into $G/H = SL(3,\mathbb{R})/SO(3) \times SL(2,\mathbb{R})/SO(2)$. In this case the threshold will be given by the sum of a order-3/2 $SL(3,\mathbb{Z})$ and a order-1 $SL(2,\mathbb{Z})$ series. The seven-dimensional threshold will be shown to be given by a order-3/2 $SL(5,\mathbb{Z})$ series.

The structure of this paper is as follows. In Section 2 we briefly review the situation in ten dimensions. In Section 3 we compactify to eight dimensions, calculate the perturbative contributions corresponding to the fundamental $(1,0)$ string, and then generalize to (p,q)-strings. In Section 4 we make the invariance the result in Section 3 under U-duality manifest, and show the presence of a logarithmic correction overlooked by our argument in Section 3. In Section 5 we compactify to lower dimensions, calculate the seven-dimensional threshold and consider briefly (and incompletely) the six-dimensional case. Finally, Section 6 contains our conclusions. In Appendix A we present some useful formulae on the expansion and regularization of $SL(3,\mathbb{Z})$ Eisenstein series.

2 The IIB string in ten dimensions

The lowest-order bosonic effective action of the type IIB string in ten dimensions in the Einstein frame is [5]

$$S_{10}^E = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_E} \left[R - \frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\tau_2^2} - \frac{1}{12\tau_2}(\tau H_N + H_R)_{\mu\nu\rho}(\bar{\tau} H_N + H_R)^{\mu\nu\rho} \right] \tag{2.1}$$

with $\kappa_{10}^2 = 2^6 \cdot \pi^7 \alpha'^4$*. We have set the self-dual four-form to zero since it will not play any role in the subsequent discussion. The complex scalar τ contains the dilaton (string coupling constant) as well as the Ramond-Ramond axion:

$$\tau = a + i e^{-\phi} \tag{2.2}$$

while as usual

$$H_{\mu\nu\rho}^{\alpha} = \partial_\mu B_{\nu\rho}^{\alpha} + \text{cyclic} \tag{2.3}$$

*From now on we set $\alpha' = 1$.

is the field strength of the R-R ($\alpha = 1$ or R) or NS-NS ($\alpha = 2$ or N) antisymmetric tensor. Transforming to the string σ-model frame $g_E = e^{-\phi/2} g_\sigma$ we obtain

$$S_{10}^\sigma = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_\sigma} \left[e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_N^2 \right) - \frac{1}{2}(\partial a)^2 - \frac{1}{12}(aH_N + H_R)^2 \right] \quad (2.4)$$

The NS-NS fields have the usual tree-level dilaton dependence (the string coupling is $g = e^\phi$), while the R-R couplings have no dilaton dependence at tree level.

The effective action is invariant under an $SL(2, \mathbb{R})_\tau$ symmetry

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1 \quad (2.5)$$

$$\begin{pmatrix} B_N \\ B_R \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_N \\ B_R \end{pmatrix} \quad (2.6)$$

while the Einstein metric and the self-dual four-form are invariant. A discrete subgroup of this symmetry, $SL(2, \mathbb{Z})_\tau$, is conjectured to be an exact non-perturbative symmetry of the IIB string. It acts as $PSL(2, \mathbb{Z})$ on the complex scalar τ plus the charge-conjugation symmetry $\tau \to \tau$, $B^\alpha \to -B^\alpha$. Various arguments for this symmetry have been given [6], including the construction of the D1 and D3 branes as well as all the (p, q) strings.

Because of the large supersymmetry, the leading terms in the effective action that have non-zero quantum corrections are terms with eight derivatives including $t_8 t_8 R^4$, $\epsilon_{10}\epsilon_{10} R^4$ and $\epsilon_{10} t_8 B_N R^4$. The tensor t_8 is the standard eight-index tensor arising in string amplitudes, while ϵ_{10} is the ten-dimensional Levi-Civita tensor†. In the case of N=1 ten-dimensional supersymmetry it was shown that these terms must combine into two different superinvariants [7] whose bosonic parts are

$$J_0 = t_8 t_8 R^4 + \frac{1}{8} \epsilon_{10}\epsilon_{10} R^4 \ , \quad J_1 = t_8 t_8 R^4 - \frac{1}{4} \epsilon_{10} t_8 B_N R^4 \quad (2.7)$$

We shall assume that this still holds for the case of N=2 supersymmetry in ten dimensions, and this is certainly in agreement with our results. In particular, J_1 contains a CP-odd anomaly related coupling and is therefore believed to receive only one-loop corrections. The J_0 invariant is expected not to receive perturbative corrections beyond one loop but it is not protected from non-perturbative corrections. Indeed it will be the purpose of this paper to obtain an exact non-perturbative result for this coupling.

We will now describe the R^4 couplings in the two ten-dimensional type II string theories in the hope of clarifying some confusion in the literature‡. At tree level the effective action for these terms has been calculated from S-matrix and σ-model computations [8] to be:

$$S_{R^4}^{\text{tree}} = 2\zeta(3) \mathcal{N}_{10} \int d^{10}x \sqrt{-g_\sigma} \ \tau_2^2 \ \left(t_8 t_8 + \frac{1}{8} \epsilon_{10}\epsilon_{10} \right) R^4 \quad (2.8)$$

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†We use the same normalization as in [10, 11], namely $t_8 F^4 := t^{\alpha_1\alpha_2\ldots\alpha_8} F_{\alpha_1\alpha_2} \ldots F_{\alpha_7\alpha_8} = 24F^4 - 6(F^2)^2$ for any antisymmetric matrix F, omitting the Levi-Civita tensor of [10]. In particular, $\epsilon_{10} t_8 R^4 = 24 \operatorname{Tr} R^4 - 6(\operatorname{Tr} R^2)^2$ and $t_8 t_8 R^4 = 24 t_8 \operatorname{Tr} R^4 - 6 t_8 \operatorname{Tr} R^2 \operatorname{Tr} R^2 = 12(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})^2 + \ldots$ The expression $\epsilon_{10}\epsilon_{10} R^4 = -96(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})^2 + \ldots$ is the continuation to ten dimensions of twice the eight-dimensional Euler density $2\epsilon_8\epsilon_8 R^4$. We take the spacetime to be minkowskian.

‡We would like to thank A. Tseytlin for sharing his insights on the issue.

$$= 2\zeta(3)\mathcal{N}_{10} \int d^{10}x\sqrt{-g_E}\ \tau_2^{\frac{3}{2}}\ \left(t_8 t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4$$

in the σ and Einstein frames, respectively. The coefficient is given by

$$\mathcal{N}_{10} = \frac{1}{3 \cdot 2^8 \kappa_{10}^2} \tag{2.9}$$

This result holds both for type IIA and IIB ten-dimensional superstrings, and does not involve the CP-odd J_1 coupling. Since in the type II string theory supersymmetry is respected by the loop expansion, this confirms the statement that J_0 is still a superinvariant for $N = 2$ ten-dimensional supersymmetry.

At one-loop one finds a non-zero CP-even contribution

$$S_{R^4}^{1-\text{loop}} = \frac{2\pi^2}{3}\mathcal{N}_{10} \int d^{10}x\sqrt{-g_\sigma}\ \left(t_8 t_8 \pm \frac{1}{8}\epsilon_{10}\epsilon_{10}\right) R^4 \tag{2.10}$$

where the $+$ sign (resp. $-$) occurs for IIB (resp. IIA) theories. There is furthermore a one-loop contribution to the CP-odd $\epsilon_{10}t_8R^4$ term in the type IIA theory. Both these results can be inferred from our eight-dimensional calculation in section 3.2 but we have checked them also directly in ten dimensions. The one-loop IIB threshold therefore reduces to the J_0 invariant, while the type IIA threshold involves a combination $2J_1 - J_0$ of the two invariants. This is compatible with the D=11 supergravity limit of type IIA, while the vanishing of $\epsilon_{10}t_8R^4$ coupling is compatible with the type IIB $SL(2,\mathbb{Z})_\tau$ symmetry. Further consistency checks are obtained under compactification of IIA/B superstring on $K3$. The reduction of J_0 on $K3$ is zero, so that Eq. (2.9) implies the absence of tree-level R^2 coupling in $N = 4$ type II vacua. Eq. (2.10) further implies that there are no one-loop R^2 corrections in six-dimensional IIB/$K3$ superstring, while the R^2 threshold is equal to the Euler number of $K3$ in the type IIA case. Moreover one-loop $B \wedge R \wedge R$ terms have to occur only in six-dimensional type IIA theory, as found by an explicit calculation [14].

It is strongly suspected that there are no further perturbative contributions to these terms. On the other hand, in the type IIB theory there can be non-perturbative contributions due to the D-instantons [3]. Indeed, the perturbative result is not invariant under the $SL(2,\mathbb{Z})_\tau$ symmetry. In [3] the exact non-perturbative threshold for the $t_8t_8R^4$ term (or more accurately for the J_0 superinvariant) was conjectured, by covariantizing the perturbative result under the $SL(2,\mathbb{Z})_\tau$ symmetry:

$$S_{R^4} = \mathcal{N}_{10} \int d^{10}x\sqrt{-g_E}\ f_{10}(\tau,\bar\tau)\ (t_8 t_8\ R^4 + \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4) \tag{2.11}$$

where

$$f_{10}(\tau,\bar\tau) = \hat{\sum_{m,n\in \mathrm{Z}}} \frac{\tau_2^{3/2}}{|m+n\tau|^3} \tag{2.12}$$

A hat over a sum indicates omission of the (0,0) term. f_{10} is manifestly $SL(2,\mathbb{Z})$-invariant. It can be expanded at large τ_2 to reveal the perturbative and D-instanton contributions

$$f_{10} = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\frac{1}{\sqrt{\tau_2}} + 8\pi\sqrt{\tau_2}\sum_{m\neq 0}\sum_{n=1}^{\infty} e^{2\pi i m n \tau_1}\left|\frac{m}{n}\right| K_1(2\pi|m|n\tau_2) \tag{2.13}$$

K_1 is the K Bessel function defined in the Appendix. The first term is the tree-level term, the second one corresponds to the one-loop correction, while the rest are exponentially suppressed as $e^{-1/g}$ at weak coupling and come from D-instantons. The threshold for a circle compactification of type IIB was further obtained in [17]. The one for the type IIA theory follows from $R \to 1/R$ duality.

We finally mention that the results above imply the following terms in the D=11 M-theory effective action

$$\delta S_{11} \sim \int d^{11}x \left[\sqrt{-g}(t_8 t_8 - \frac{1}{4\cdot 3!}\epsilon_{11}\epsilon_{11})R^4 - \frac{1}{4}\epsilon_{11}t_8 C R^4\right] \tag{2.14}$$

where C is now the three-form gauge potential of D=11 supergravity. Upon compactification to ten dimensions this reproduces the one-loop terms of the IIA string. The tree level term cannot be seen in the eleven-dimensional limit. The reason is that it is produced in the ten-dimensional theory by integrating out the massive modes of the perturbative IIA string. In the decompactification limit, these masses become infinite and this term disappears.

3 The type IIB string in eight dimensions

3.1 Scalar manifold and U duality

We will now compactify the type IIB string on a two-torus. The effective tree-level action can be obtained by the standard Kaluza-Klein reduction. The scalar fields in eight dimensions are, in addition to the complex scalar τ, the two-dimensional σ-frame metric of the two-torus

$$G_{IJ} = \frac{V}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \tag{3.1}$$

written in terms of the volume $V = \sqrt{G}$ and of the complex structure modulus U of the two-torus, together with the two scalars coming from the two antisymmetric tensor backgrounds:

$$B^{\alpha}_{IJ} = \epsilon_{IJ} B^{\alpha} \tag{3.2}$$

The (physical) volume of the two-torus is $(2\pi)^2 V$. In eight dimensions, in addition to the ten-dimensional $SL(2,\mathbb{Z})_\tau$ symmetry we also expect the usual T-duality symmetry $SL(2,\mathbb{Z})_T$ acting on the T-modulus

$$T = B_N + iV \tag{3.3}$$

as well as $SL(2,\mathbb{Z})_U$ acting on the U modulus. Moreover the exchange T$\leftrightarrow$U maps the IIB to the type IIA string.

We omit the calculational details and present the Einstein-Hilbert action and scalar kinetic terms in the eight-dimensional Einstein frame:

$$S_8^E = \frac{1}{2\kappa_8^2}\int d^8x\sqrt{-g_E}\left[R - \frac{1}{6}\frac{\partial\nu^2}{\nu^2} - \frac{1}{2}\frac{\partial U\partial\bar{U}}{U_2^2} - \frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\tau_2^2} - \nu\frac{|\tau\partial B_N + \partial B_R|^2}{2\tau_2} + \cdots\right] \tag{3.4}$$

where $\kappa_8 = \kappa_{10}/2\pi$.The scalar

$$\nu = \frac{1}{\tau_2 V^2} \tag{3.5}$$

is invariant under the $SL(2,\mathbb{Z})_\tau$ duality. The above action is manifestly invariant under the $SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_U$ part of the duality. The $SL(2,\mathbb{Z})_T$ duality becomes manifest (and $SL(2,\mathbb{Z})_\tau$ duality hidden) if we introduce the T variable (3.3) together with the T-duality invariant eight-dimensional dilaton

$$e^\lambda = \frac{1}{V\tau_2^2} \tag{3.6}$$

which is the standard eight-dimensional string expansion parameter, and with the complex scalar

$$\xi = -B_R + iaV \tag{3.7}$$

The action can now be written as

$$S_8^E = \frac{1}{2\kappa_8^2}\int d^8x\sqrt{-g_E}\left[R - \frac{1}{6}(\partial\lambda)^2 - \frac{1}{2}\frac{\partial U\partial\bar{U}}{U_2^2} - \frac{1}{2}\frac{\partial T\partial\bar{T}}{T_2^2} - \right. \tag{3.8}$$

$$\left. -\frac{e^\lambda}{2T_2}\left|\partial\left(\frac{\mathrm{Im}(T\bar{\xi})}{T_2}\right) + T\partial\left(\frac{\mathrm{Im}\xi}{T_2}\right)\right|^2\right]$$

Under an $SL(2,\mathbb{Z})_T$ transformation

$$T \to \frac{aT+b}{cT+d} \quad , \quad \xi \to \frac{\xi}{cT+d} \tag{3.9}$$

the rest being inert, the effective action is indeed invariant. The field ξ transforms as the complex coordinate on a torus of complex structure T. The maximal invariance of the action in Eq. (3.4) is known to be larger, namely $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ [12]. Its discrete subgroup $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})_U$ was conjectured [13] to be an exact symmetry (U-duality) of the eight-dimensional IIB string. The symmetry becomes manifest if we introduce the following symmetric matrix with determinant one:

$$M = \nu^{1/3}\begin{pmatrix} \frac{1}{\tau_2} & \frac{\tau_1}{\tau_2} & \frac{\mathrm{Re}(B)}{\tau_2} \\ \frac{\tau_1}{\tau_2} & \frac{|\tau|^2}{\tau_2} & \frac{\mathrm{Re}(\bar{\tau}B)}{\tau_2} \\ \frac{\mathrm{Re}(B)}{\tau_2} & \frac{\mathrm{Re}(\bar{\tau}B)}{\tau_2} & \frac{1}{\nu} + \frac{|B|^2}{\tau_2} \end{pmatrix} \quad , \quad M = M^T \quad , \quad \det(M) = 1 \tag{3.10}$$

where we introduced the complex scalar $B = B_R + \tau B_N$ that transforms the same way as ξ under $SL(2,\mathbb{Z})_\tau$ duality. The matrix M parametrizes the $SL(3,\mathbb{R})/SO(3)$ coset. In terms of M the effective action can be written as

$$S_8^E = \frac{1}{2\kappa_8^2}\int d^8x\sqrt{-g_E}\left[R - \frac{1}{2}\frac{\partial U\partial\bar{U}}{U_2^2} + \frac{1}{4}\mathrm{Tr}(\partial M\partial M^{-1})\right] \tag{3.11}$$

which shows that the scalar manifold is $SL(3,\mathbb{R})/SO(3) \times SL(2,\mathbb{R})/SO(2)$. An element Ω of $SL(3,\mathbb{R})$ acts on M as $M \to \Omega M \Omega^T$. The conjectured $SL(3,\mathbb{Z})$ part of the U-duality symmetry is generated by matrices Ω with integer entries. This symmetry can be obtained by intertwining $SL(2,\mathbb{Z})_\tau$ and $SL(2,\mathbb{Z})_T$ transformations, which are embedded in $SL(3,\mathbb{Z})$ as follows:

$$SL(2,\mathbb{Z})_\tau \ : \ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad SL(2,\mathbb{Z})_T \ : \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \tag{3.12}$$

The $SL(3,\mathbb{Z})$ part of the U-duality symmetry is easily understood in the dual IIA theory. This is M-theory compactified on a three-torus with metric G_3. The volume $\sqrt{G_3}$ together with the three-index antisymmetric tensor C_{123} corresponds to the type IIB modulus U. The remaining metric with unit determinant $\tilde{G}_3 = \det(G)^{-1/3} G_3$ corresponds to the IIB $SL(3,\mathbb{Z})$ matrix M, and $SL(3,\mathbb{Z})$ is the modular group of the 3-torus.

3.2 Perturbative gravitational thresholds

We will parametrize the eight-dimensional threshold corrections as

$$S_8^{R^4} = \mathcal{N}_8 \int d^8x \sqrt{-g_E} \left[(\Delta_{tt} t_8 t_8 + \frac{1}{4} \Delta_{\epsilon\epsilon} \epsilon_8 \epsilon_8 + \Theta\ t_8 \epsilon_8) R^4 \right] \tag{3.13}$$

where $\mathcal{N}_8 = (2\pi)^2 \mathcal{N}_{10}$ and we are in the eight-dimensional Einstein frame (note that $\Delta_{tt}, \Delta_{\epsilon\epsilon}$ and Θ are all dimensionless in eight dimensions). The ten-dimensional result discussed in section 2 implies the following large-volume behaviour of the thresholds

$$\lim_{T_2 \to \infty} \frac{\Delta_{tt}}{T_2} = \lim_{T_2 \to \infty} \frac{\Delta_{\epsilon\epsilon}}{T_2} = \sqrt{\tau_2}\ f_{10}(\tau, \bar{\tau})\ ,\quad \lim_{T_2 \to \infty} \frac{\Theta}{T_2} = 0 \tag{3.14}$$

The tree-level result is obtained directly by compactification of the ten-dimensional result:

$$\Delta_{tt}^{\text{tree}} = 2\zeta(3) V \tau_2^2 = 2\zeta(3) \frac{T_2^{3/2}}{\nu^{1/2}} \tag{3.15}$$

The one-loop result can also be directly computed by evaluating the one-loop scattering amplitude of four gravitons together with one modulus field of T^2, using the same techniques as for the four-dimensional R^2 terms [14]. The one-loop R^4 thresholds are also IR-divergent, and can be regularized by introducing an IR cutoff by hand. § This introduces an ambiguity in the moduli independent part of the threshold.

We therefore compute the following amplitude:

$$\mathcal{A} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \prod_{i=1}^{4} \langle \int \frac{\mathrm{d}^2 z_i}{\pi} \epsilon^i_{\bar{\alpha}_i \alpha_i} V^{\bar{\alpha}_i \alpha_i}(p_i, \bar{z}_i, z_i) \int \frac{\mathrm{d}^2 z_5}{\pi} V_\phi(p_5, \bar{z}_5, z_5) \rangle \tag{3.16}$$

In this expression, $\epsilon^i_{\alpha\beta}$ denote the transverse symmetric traceless polarization tensors of the four gravitons, whose vertex operators in the zero ghost picture read

$$V^{\bar{\alpha}\alpha}(p, \bar{z}, z) = [\bar{\partial} X^{\bar{\alpha}}(\bar{z}, z) + ip \cdot \bar{\psi}(\bar{z}) \bar{\psi}^{\bar{\alpha}}(\bar{z})][\partial X^{\alpha}(\bar{z}, z) + ip \cdot \psi(z) \psi^{\alpha}(z)] \mathrm{e}^{ip \cdot X(\bar{z}, z)} \tag{3.17}$$

The modulus field vertex operator is defined in a similar way:

$$V_\phi(p, \bar{z}, z) = v_{IJ}(\phi) [\bar{\partial} X^I(\bar{z}, z) + ip \cdot \bar{\psi}(\bar{z}) \bar{\psi}^I(\bar{z})][\partial X^J(\bar{z}, z) + ip \cdot \psi(z) \psi^J(z)] \mathrm{e}^{ip \cdot X(\bar{z}, z)} \tag{3.18}$$

§This regularization breaks modular invariance. A stringy IR regularization method was developped in [15], but in this case it breaks the supersymmetry. In the cases where it can be applied, it agrees with a the usual non-modular invariant regularization.

in terms of the metric and antisymmetric fields in the internal T^2 directions:

$$v_{IJ}(\phi) = \partial_\phi (G_{IJ} + B_{IJ}) \ , \quad I,J = 1,2 \tag{3.19}$$

The correlators are evaluated in the partition function of the toroidally compactified superstring

$$Z = \frac{1}{4} \sum_{\bar{a},\bar{b}=0}^{1} (-)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}} \vartheta \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}^4 (-)^{a+b+ab} \vartheta \begin{bmatrix} a \\ b \end{bmatrix}^4 \frac{\Gamma_{2,2}(T,U)}{\tau_2^3 |\eta|^{24}} \tag{3.20}$$

where μ distinguishes between type IIA ($\mu = 0$) and type IIB ($\mu = 1$) string theories. $\Gamma_{2,2}$ is the usual $(2,2)$ lattice sum describing the possible wrappings of the string world-sheet on the target space torus. Finally the correlation function has to be integrated over the positions z_i of the vertices on the world-sheet, and over the world-sheet Teichmüller parameter τ (we use the same notation as for the scalar modulus τ, but the context should make clear which one is meant), on the usual fundamental domain $\mathcal{F}$ of the $SL(2,\mathbb{Z})$ modular group. The different spin structures labelled by $a, b, \bar{a}, \bar{b}$ make distinct contributions to the amplitude and have to be treated separately according to their parity $(-)^{ab}$.

• For even left and right moving spin structures, it can be checked from Riemann summation identity that terms with less that four fermionic contractions on both sides vanish after summing over even spin structures. This yields four powers of momenta on each side, and since we are interested in the leading $\mathcal{O}(p^8)$ contribution we can set $e^{ip\cdot x} = 1$. The corresponding eight fermions on each side have to be provided by the four gravitons vertex operators 3.17, while only the bosonic part of 3.18 can contribute.

The four pairs of fermions can be contracted in 60 different ways, each of these giving, up to $g^{\mu\nu}$ factors, a contribution

$$\frac{1}{2} \sum_{a,b \text{ even}} (-)^{a+b} \vartheta^4 \begin{bmatrix} a \\ b \end{bmatrix} \prod_{i}^{4} \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (Z_i) \vartheta_1'(0)}{\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (0) \vartheta_1(Z_i)} = -\frac{1}{2} (2\pi\eta^3)^4 \tag{3.21}$$

where the result, after spin structure summation, no longer depends on the contractions Z_i. On the other hand, the ϕ modulus insertion yields a derivative with respect to ϕ:

$$\langle V_\phi(\bar{z}_5, z_5) \rangle = \frac{1}{\pi\tau_2} \partial_\phi \Gamma_{2,2} \tag{3.22}$$

again independent of z^5. Integrating over the positions of the vertices according to $\int \frac{\mathrm{d}^2 z}{\tau_2} = 1$, we obtain

$$\mathcal{A}_{\bar{e}-e} = \mathcal{T}_{\bar{e},e} \int \frac{\mathrm{d}^2\tau}{\tau_2^2} \frac{1}{4} \frac{1}{\tau_2^3} \left(\frac{\tau_2}{\pi} \right)^5 \frac{(2\pi\eta^3)^4 (2\pi\bar{\eta}^3)^4}{|\eta|^{24}} \frac{1}{\pi\tau_2} \partial_\phi \Gamma_{2,2} \tag{3.23}$$

where the kinematical factor $\mathcal{T}_{\bar{e},e}$ is provided by the fermionic contractions on each side:

$$\mathcal{T}_{\bar{e},e} = 4 \ t_8^{\bar{\alpha}_1\bar{\beta}_1\ldots\bar{\alpha}_4\bar{\beta}_4} t_8^{\alpha_1\beta_1\ldots\alpha_4\beta_4} \epsilon^1_{\bar{\alpha}_1\alpha_1} p^1_{\bar{\beta}_1} p^1_{\beta_1} \ldots \epsilon^4_{\bar{\alpha}_4\beta_4} p^4_{\bar{\beta}_4} p^4_{\beta_4} \tag{3.24}$$

We now use the momentum space representation of the Riemann tensor for a gravitational fluctuation around a flat background

$$R_{\bar{\alpha}\bar{\beta}\alpha\beta} = \frac{1}{2} \left[\epsilon_{\bar{\alpha}\alpha} p_{\bar{\beta}} p_\beta - (\alpha \leftrightarrow \beta) - (\bar{\alpha} \leftrightarrow \bar{\beta}) + ((\alpha, \bar{\alpha}) \leftrightarrow (\beta, \bar{\beta})) \right] \tag{3.25}$$

together with the symmetries ($\alpha_i \leftrightarrow \beta_i$) of the t_8 tensor, to rewrite

$$\mathcal{T}_{\bar{e},e} = 16\ t_8^{\bar\alpha_1\bar\beta_1\ldots\bar\alpha_4\bar\beta_4} t_8^{\alpha_1\beta_1\ldots\alpha_4\beta_4} R_{\bar\alpha_1\bar\beta_1\alpha_1\beta_1}\ldots R_{\bar\alpha_4\bar\beta_4\alpha_4\beta_4} := 16\ t_8 t_8 R^4 \tag{3.26}$$

• When one left or right spin structure is odd, the computation is significantly different, since one vertex, say the modulus one, has to be converted to the (-1) ghost picture and supplemented by an insertion of a supercurrent. Effectively, in the $\overline{\text{odd}}-\text{odd}$ case,

$$V_\phi \to v_{IJ}\bar\psi(\bar z)^I\psi(z)^J \mathrm{e}^{ip\cdot X(\bar z,z)}\ G_F(0)\bar G_F(0) \tag{3.27}$$

where $G_F = \partial X^\mu\psi_\mu + G_{KL}\partial X^K\psi^L$. The ten fermionic zero modes on both sides then have to be saturated by the eight fermions in the graviton vertices together with the two from the modulus vertex and the supercurrent, while the integral over the fermionic non-zero-modes induces the replacement $\vartheta\left[{1 \atop 1}\right]^4 \to (2\pi\eta^3)^4$ in the partition function (3.20). The modulus insertion can be converted into a derivative with respect to ϕ, thanks to the supersymmetric partner of identity (3.22):

$$\langle v_{IJ}\bar\psi^I\psi^J G_{KL}\bar\partial X^K\bar\psi^L G_{MN}\partial X^M\psi^N\rangle = \frac{\sigma_\phi}{\pi\tau_2}\partial_\phi\Gamma_{2,2} \tag{3.28}$$

where $\sigma_\phi = 1$ for the $T,\overline{T}$ moduli and $\sigma_\phi = -1$ for the $U,\overline{U}$ moduli. The $(\bar o, o)$ contribution therefore reduces to the same contribution as the $(\bar e, e)$ one, but for a kinematical structure

$$\mathcal{T}_{\bar o,o} = \epsilon_8^{\bar\alpha_1\bar\beta_1\ldots\bar\alpha_4\bar\beta_4}\epsilon_8^{\alpha_1\beta_1\ldots\alpha_4\beta_4}\epsilon^1_{\bar\alpha_1\alpha_1}p^1_{\bar\beta_1}p^1_{\beta_1}\ldots\epsilon^4_{\bar\alpha_4\alpha_4}p^4_{\bar\beta_4}p^4_{\beta_4} = 4\ \epsilon_8\epsilon_8 R^4 \tag{3.29}$$

and a crucial sign $(-)^{\mu+\sigma_\phi}$ depending on both the modulus and the superstring we are considering. The fact that the eight-dimensional Euler density $\epsilon_8\epsilon_8 R^4$ is a total derivative should cause no concern here, since this property is lost when its coefficient becomes moduli-dependent. Note also that in contrast to the four-dimensional case [14], $t_8 t_8 R^4$ is now distinct from $\epsilon_8\epsilon_8 R^4$, so that there cannot be any interferences between $\bar e - e$ and $\bar o - o$ amplitudes.

• When the spin structure is odd on one side and even on the other side, the modulus vertex has to be chosen in the (-1,0) picture. The considerations of the previous cases apply on each side, and one can again convert the modulus insertion into a derivative thanks to

$$\begin{aligned}\langle v_{IJ}\bar\psi^I\partial X^J G_{KL}\bar\partial X^K\bar\psi^L\rangle &= i\sigma_\phi\chi_\phi\partial_\phi\Gamma_{2,2}\\ \langle v_{IJ}\bar\partial X^I\psi^J G_{MN}\partial X^M\psi^N\rangle &= i\chi_\phi\partial_\phi\Gamma_{2,2}\end{aligned} \tag{3.30}$$

where χ_ϕ distinguishes chiral moduli ($\chi_T = \chi_U = 1$) from antichiral ones ($\chi_{\overline{T}} = \chi_{\overline{U}} = -1$). The result is again the same as in the $\bar e - e$ case, but for a kinematical coefficient

$$8\ \epsilon_8^{\bar\alpha_1\bar\beta_1\ldots\bar\alpha_4\bar\beta_4}\epsilon_8^{\alpha_1\beta_1\ldots\alpha_4\beta_4} R_{\bar\alpha_1\bar\beta_1\alpha_1\beta_1}\ldots R_{\bar\alpha_4\bar\beta_4\alpha_4\beta_4} := 8\ t_8\epsilon_8 R^4 \tag{3.31}$$

and a prefactor $i(-1)^\mu$ (resp. $i(-1)^{\mu+\sigma_\phi}$) for the $\bar e - o$ (resp. $\bar o - e$) cases.

Putting all contributions together, we can write the scattering amplitude as

$$\mathcal{A} = 48\pi\left(t_8 + \frac{i}{2}(-)^{\mu+\sigma_\phi+\chi_\phi}\epsilon_8\right)\left(t_8 + \frac{i}{2}(-)^{\chi_\phi}\epsilon_8\right)\partial_\phi\int\frac{\mathrm{d}^2\tau}{\tau_2}\Gamma_{2,2}(T,U) \tag{3.32}$$

where we fitted the overall coefficient to obtain the correct decompactification limit. The remaining fundamental domain integral was evaluated long ago in [16]:

$$\int \frac{d^2\tau}{\tau_2}\Gamma_{2,2}(T,U) = -\log(T_2|\eta(T)|^4 U_2|\eta(U)|^4) \tag{3.33}$$

up to an irrelevant moduli-independent infrared ambiguity.

It is straightforward to integrate Eq. (3.32) with respect to the moduli ϕ to obtain the one-loop corrections to the CP-even $t_8t_8R^4$ and $\epsilon_8\epsilon_8R^4$ couplings. In the type IIB case, we obtain:

$$\begin{aligned}\mathcal{S}^{CPeven}_{1-loop} = -2\pi \int d^8x\sqrt{-g_\sigma}\Big(\Big[\log(T_2|\eta(T)|^4) + \log(U_2|\eta(U)|^4) t_8t_8R^4\Big] \\ -\frac{1}{4}\Big[\log(T_2|\eta(T)|^4) - \log(U_2|\eta(U)|^4)\epsilon_8\epsilon_8R^4\Big]\Big)\end{aligned} \tag{3.34}$$

However, in the CP-odd case, only the harmonic part of Eq. (3.33) can be integrated in the form of a moduli-dependent $t_8\epsilon_8R^4$ coupling, while the non-harmonic part $\log T_2U_2$, imputable to the IR divergence, has to be treated separately. Let X_7 be the Chern-Simons form associated to the closed eight form $t_8\epsilon_8R^4 = 24\,\mathrm{Tr}\,R^4 - 6(\,\mathrm{Tr}\,R^2)^2$: we can then rewrite the CP-odd coupling of four gravitons and T^2 moduli as (in the type IIB case)

$$\mathcal{S}^{CPodd}_{1-loop} = 4\pi\int d^8x\sqrt{-g_\sigma}\,\mathrm{Im}\,\Big[\log\eta^4(U)\Big]\ \epsilon_8t_8R^4 - 4\pi\int d^8x\frac{1}{U_2}X_7\wedge dU_1 \tag{3.35}$$

The type IIA case is obtained under $T \leftrightarrow U$ exchange.

Going to the Einstein frame only modifies this result by higher derivative coupling to the dilaton. We conclude that the tree-level and one-loop corrections to the R^4 thresholds for the eight-dimensional type IIB string can be written as:

$$\Delta_{tt} = 2\zeta(3)V\tau_2^2 - 2\pi\log(T_2|\eta(T)|^4) - 2\pi\log(U_2|\eta(U)|^4) \tag{3.36}$$

$$\Delta_{\epsilon\epsilon} = 2\zeta(3)V\tau_2^2 - 2\pi\log(T_2|\eta(T)|^4) + 2\pi\log(U_2|\eta(U)|^4) \tag{3.37}$$

$$\Theta = 4\pi\,\mathrm{Im}\,[\log\eta(U)^4] \tag{3.38}$$

In particular, the contribution of the world-sheet instantons of the fundamendal IIB string to Δ_{tt} is given by

$$\mathcal{I}_{1,0} = -8\pi\mathrm{Re}\ \log\left[\prod_{n=1}^{\infty}\left(1-e^{2\pi i nT}\right)\right] \tag{3.39}$$

3.3 Non-perturbative (p,q)-string instanton contribution

In addition to the fundamental string, the ten-dimensional type IIB superstring theory possesses solitonic objects of various dimensions. Wrapped on the compactification manifold, these configurations yield instantons in lower dimensions. These instantons preserve one half of the ten-dimensional supersymmetry, and therefore have the correct number of fermionic

zero-modes to contribute to R^4 couplings. The D-instanton is localized to a point in space-time and yields the same contribution in any compactification of type IIB theory, up to a volume factor. The D3,5,7-branes and the NS 5-brane contribute for $D \leq 6,4,2,4$ respectively. On the other hand, the D-strings start contributing for $D=8$, where they can supersymmetrically wrap around the two-torus. The D-strings have charges (p,q) under the NS-NS and R-R antisymmetric tensors (p,q are coprime: $(p,q)=1$), and form a multiplet under $SL(2,\mathbb{Z})_\tau$ symmetry. The $(1,0)$ string corresponds to the fundamental type IIB string, and its contribution is known from the perturbative one-loop computation (3.39). One can then apply $SL(2,\mathbb{Z})_\tau$ to infer the contributions of all (p,q) strings.

The world-sheet Nambu-Goto action of a (0,1) D-string is known to be [18]:

$$S_{0,1} = \frac{e^{-\phi}}{2\pi}\int d^2\sigma\sqrt{\det(\hat{G}+\mathcal{F})} + \frac{i}{2\pi}\int \hat{B}_R \tag{3.40}$$

for vanishing background Ramond scalar expectation value. The hat denotes pulled-back quantities: $\hat{G}_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$, etc. and $\mathcal{F} = F - \hat{B}_N$, with $F_{\alpha\beta}$ the field strength of the world-sheet gauge field. When the background scalar $a=\tau_1$ is switched on, this becomes

$$S_{0,1} = \frac{|\tau|}{2\pi}\int d^2\sigma\sqrt{\det(\hat{G}+\mathcal{F})} + \frac{i}{2\pi}\int \left(\hat{B}_R + \tau_1\mathcal{F}\right) \tag{3.41}$$

where the $\tau_1\mathcal{F}$ coupling ensures anomaly cancellation [19].

Using Cartesian coordinates $X^1, X^2 \in [0,2\pi]$ for the target space torus and $\sigma_{1,2}\in[0,2\pi]$ for the D1-brane, the σ-frame target metric is given in Eq. (3.1). The supersymmetric embedding wrapping the string world-sheet around the two-torus can be written as

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \tag{3.42}$$

A non-degenerate orientation preserving mapping is obtained $N = m_1 n_2 - m_2 n_1 > 0$, while for $m_1 n_2 - m_2 n_1 < 0$, the orientation is reversed and the induced complex structure is complex-conjugated. The equations of motion also require $\mathcal{F}$ =constant. Setting the constant to zero correcponds to the (0,1) string [20]. Evaluating the action on this instanton configuration, we obtain

$$S^{\rm class}_{0,1} = 2\pi|N||\tau|T_2 + 2\pi i N B_R \tag{3.43}$$

This implies that the $(0,1)$-string instanton has an effective T-modulus given by

$$T_{0,1} = B_R + i|\tau|T_2 \tag{3.44}$$

Using the SL(2,$\mathbb{Z}$) symmetry we obtain that the effective modulus of a (p,q) string is

$$T_{p,q} = (qB_R - pB_N) + i|p+q\tau|T_2 \tag{3.45}$$

Thus, the contribution from all (p,q) D-strings can be written as

$$\mathcal{I}_{p,q} = -8\pi {\rm Re}\ \log\left[\prod_{n=1}^{\infty}\left(1-e^{2\pi i n T_{p,q}}\right)\right] \tag{3.46}$$

Together with the D-instanton contribution (2.13), we obtain our conjecture for the exact R^4 thresholds in 8 dimensions:

$$\Delta_{tt} = V\sqrt{\tau_2} f_{10}(\tau,\bar\tau) - 2\pi\log T_2 - 2\pi\log(U_2|\eta(U)|^4) + \sum_{(p,q)=1} \mathcal{I}_{p,q} \tag{3.47}$$

$$\Delta_{\epsilon\epsilon} = V\sqrt{\tau_2} f_{10}(\tau,\bar\tau) - 2\pi\log T_2 + 2\pi\log(U_2|\eta(U)|^4) + \sum_{(p,q)=1} \mathcal{I}_{p,q} \tag{3.48}$$

while we do not expect any corrections to the already duality-invariant $t_8\epsilon_8 R^4$ coupling. In order to be acceptable, this result should satisfy the requirement of $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})_U$ invariance. The invariance under $SL(2,\mathbb{Z})_U$ is already incorporated in the above Equations. Indeed, $\log(U_2|\eta(U)|^4$ is the order-1 Eisenstein series for $SL(2,\mathbb{Z})_U$, and the other terms are invariant under that group. We will show in the next Section the remaining terms can be rewritten in terms of the $SL(3,\mathbb{Z})$ Eisenstein order-3/2 series, which will make the invariance under $SL(3,\mathbb{Z})$ obvious. This series is actually logarithmically divergent, and it will turn out necessary to add to Eqs. (3.47,3.48) an extra logarithmic contribution not captured by perturbation theory.

4 $SL(3,\mathbb{Z})$ invariance of the R^4 thresholds

In view of the ten-dimensional result (2.12), it is natural to conjecture that the eight-dimensional $t_8t_8R^4$ threshold can be written in terms of the order-$s=3/2$ Eisenstein series for $SL(3,\mathbb{Z})$. This will fulfill the requirements of $SL(3,\mathbb{Z})$ invariance and ten-dimensional decompactification limit. In fact we shall show here that this series gives precisely the result motivated in the previous Section.

The Eisenstein $SL(3,\mathbb{Z})$ series with order-s is defined as

$$E_s \equiv \hat{\sum_{m_i\in Z}} \left(\sum_{i,j=1}^{3} m_i M^{ij} m_j\right)^{-s} = \hat{\sum_{m_i\in \mathrm{Z}}} \nu^{-s/3}\left[\frac{|m_1+m_2\tau+m_3B|^2}{\tau_2}+\frac{m_3^2}{\nu}\right]^{-s} \tag{4.1}$$

where $\hat{\sum}$ stands for the sum with (0,0,0) omitted. E_s is by construction $SL(3,\mathbb{Z})$-invariant.

Introducing the Laplacian operator on the $SL(3,\mathbb{R})/SO(3)$ homogeneous space

$$\Delta = 4\tau_2^2\partial_\tau\partial_{\bar\tau} + \frac{1}{\nu\tau_2}\left|\partial_{B_N}-\tau\partial_{B_R}\right|^2 + 3\partial_\nu(\nu^2\partial_\nu) \tag{4.2}$$

we deduce that E_s is an eigenfunction of the Laplacian

$$\Delta E_s = \frac{2s(2s-3)}{3}E_s \tag{4.3}$$

For $s>3/2$, E_s is an absolutely convergent series. $E_{\frac{3}{2}}$ is logarithmically divergent and is also annihilated by the Laplacian. This is the relevant function for the threshold since we already know that there is a physical logarithmic divergence in the eight-dimensional $t_8t_8R^4$ term at one-loop. It also matches the ten-dimensional conjecture [17], as well as the

recent M-theory motivated proposal [4]. In the sequel we will use ζ-function regularization by keeping s arbitrary and larger then 3/2 where the sum converges. We use an "$\overline{MS}$"-like definition:

$$\hat{E}_{\frac{3}{2}} = \lim_{\epsilon\to 0^+}\left[E_{\frac{3}{2}+\epsilon} - \frac{2\pi}{\epsilon} - 4\pi(\gamma-1)\right] \tag{4.4}$$

where γ is the Euler constant. Because of this subtraction, $\hat{E}_{\frac{3}{2}}$ is no longer a zero-mode of the Laplacian, but instead satisfies

$$\Delta\hat{E}_{\frac{3}{2}} = 4\pi \tag{4.5}$$

We now show that this function contains all the contributions expected from our previous arguments. We introduce the integral representation

$$E_s(M) = \frac{\pi^s}{\Gamma(s)}\int_0^\infty \frac{dt}{t^{1+s}} \hat{\sum_{m_i\in Z}} \exp\left[-\frac{\pi}{t}\left(\sum_{i,j=1}^{3} m_i M^{ij} m_j\right)\right] \tag{4.6}$$

$$= \nu^{-s/3}\frac{\pi^s}{\Gamma(s)}\int_0^\infty \frac{dt}{t^{1+s}} \hat{\sum_{m_i\in Z}} \exp\left[-\frac{\pi}{t}\left(\frac{m_3^2}{\nu} + \frac{|m_1+m_2\tau+m_3B|^2}{\tau_2}\right)\right]$$

In Appendix A, we evaluate this integral for arbitrary s. Here we will set $s = 3/2$ and mention at the appropriate point the modification from the regularization.

We first split the sum as $\hat{\sum}_{m_i\in Z} = \sum_{m_1\neq 0, m_2,m_3=0} + \sum_{m_1\in Z}\hat{\sum}_{m_{2,3}\in Z}$ to obtain

$$\hat{E}_{\frac{3}{2}} = 4\pi\nu^{-1/2}\sum_{m_1=1}^{\infty}\int_0^\infty \frac{dt}{t^{5/2}}\exp\left[-\frac{\pi m_1^2}{t\tau_2}\right] + J_1 = I_0 + J_1 \tag{4.7}$$

where

$$I_0 = 2\frac{\tau_2^{3/2}}{\nu^{1/2}}\zeta(3) \tag{4.8}$$

$$J_1 = 2\pi\nu^{-1/2}\int_0^\infty \frac{dt}{t^{5/2}} \hat{\sum_{m_{2,3}\in Z}}\sum_{m_1\in Z} \exp\left[-\frac{\pi}{t}\left(\frac{m_3^2}{\nu} + \frac{|m_1+m_2\tau+m_3B|^2}{\tau_2}\right)\right] \tag{4.9}$$

We now Poisson-resum on m_1, change variable $t\to 1/t$ and then separate the $m_1 = 0$ and $m_1\neq 0$ contributions to obtain $J_1 = J_2 + J_3$, with

$$J_3 = 2\pi\frac{\sqrt{\tau_2}}{\sqrt{\nu}}\int_0^\infty dt \sum_{m_1\neq 0}\hat{\sum_{m_{2,3}\in Z}} \exp\left[-\frac{\pi\tau_2 m_1^2}{t} - \frac{\pi t}{\tau_2}(m_2\tau_2+m_3B_2)^2 - \right.$$

$$\left. -\frac{\pi t m_3^2}{\nu} - 2\pi i m_1(m_2\tau_1+m_3B_1)\right] \tag{4.10}$$

where $B = B_R + \tau B_N = B_1 + iB_2$ and

$$J_2 = 2\pi\frac{\sqrt{\tau_2}}{\sqrt{\nu}}\int_0^\infty dt \hat{\sum_{m,n\in Z}} \exp\left[-\pi t\tau_2|m+nT|^2\right] \tag{4.11}$$

In the above Equation, we changed variables to the T modulus using Eq. (3.3,3.5).J_2 is in fact the piece responsible for the IR divergence. In the Appendix we show that the regulated expression is

$$\begin{aligned} J_{2,\text{reg}} &= -\pi\left(\log\tau_2 + \frac{1}{3}\log\nu\right) - 2\pi\log T_2 + \frac{2\pi^2}{3}T_2 + I_{1,0} \\ &= -\pi\left(\log\tau_2 + \frac{1}{3}\log\nu\right) - 2\pi\log\left[T_2|\eta(T)|^4\right] \end{aligned} \tag{4.12}$$

where

$$I_{1,0} = 8\pi\text{Re}\sum_{m,n=1}^{\infty}\frac{1}{n}e^{2\pi imnT} = -8\pi\text{Re }\log\left[\prod_{m=1}^{\infty}\left(1-e^{2\pi imT}\right)\right] \tag{4.13}$$

is precisely the contribution of the fundamental string world-sheet instantons.

We will now proceed to evaluate the left-over integral J_3. We will again split the summation as $(m_2,m_3)' = (m_2 \neq 0, m_3 = 0) + (m_2, m_3 \neq 0)$ and Poisson-resum over m_2 in the second sum to obtain $J_3 = I_D + J_4$ with

$$I_D = 8\pi\sqrt{\frac{\tau_2}{\nu}}\sum_{p\neq 0}\sum_{n=1}^{\infty}\left|\frac{p}{n}\right|K_1(2\pi\tau_2|p|n)e^{2\pi ipn\tau_1} \tag{4.14}$$

where K_1 is the standard Bessel function, and

$$J_4 = 2\pi\sum_{m_{1,3}\neq 0}\sum_{m_2\in Z}\frac{1}{|m_3|}\exp\left[-2\pi|m_3||m_2+m_1\tau|T_2 - 2\pi i\frac{m_3}{\tau_2}\left(m_2B_2 + m_1(\tau_1B_2 - B_1\tau_2)\right)\right] \tag{4.15}$$

The piece I_D is the contribution of the D-instantons conjectured in Ref. [17].

We now split the summation over m_2 into the $m_2 = 0$ and $m_2 \neq 0$ pieces to write $J_4 = I_{0,1} + J_5$ with

$$\mathcal{I}_{0,1} = -8\pi\text{Re }\log\left[\prod_{m=1}^{\infty}\left(1-e^{2\pi im(B_R+i|\tau|T_2)}\right)\right] \tag{4.16}$$

We recognize in $\mathcal{I}_{0,1}$ the contribution of the (0,1) D1-string instantons. Pursuing further,

$$\begin{aligned} J_5 &= 2\pi\sum_{m_i\neq 0}\frac{1}{|m_3|}\exp\left[-2\pi|m_3||m_2+m_1\tau|T_2 + 2\pi im_3\left(m_1B_R - m_2B_N\right)\right] \\ &= -4\pi Re\sum_{m_{1,2}\neq 0}\log\left(1-\exp\left[-2\pi|m_2+m_1\tau|T_2 + 2\pi i\left(m_1B_R - m_2B_N\right)\right]\right) \end{aligned} \tag{4.17}$$

where we have used the definition $B = B_R + \tau B_N$. If we denote by n the (positive) greatest common divisor from any pair of non-zero integers (m_1,m_2), then we can write $\{m_1,m_2\} = n\{p,q\}$ with $(p,q)=1$. Moreover (p,q) and $-(p,q)$ give the same contribution. Summing over (p,q) modulo this charge conjugation, we finally obtain

$$J_5 = \sum_{p,q\neq 0,(p,q)=1}\mathcal{I}_{p,q} \tag{4.18}$$

with

$$\mathcal{I}_{p,q} = -8\pi \mathrm{Re}\log\left[\prod_{n=1}^{\infty}\left(1-e^{2\pi i n T_{p,q}}\right)\right] \tag{4.19}$$

and

$$T_{p,q} = (qB_R - pB_N) + i|p+q\tau|T_2 \tag{4.20}$$

Those are recognized as the contributions of the dyonic (p,q)-string instantons (3.46).

Putting everything together we obtain the following expansion for the $SL(3,\mathbb{Z})$ Eisenstein order-3/2 series:

$$\hat{E}_{\frac{3}{2}} = 2\frac{\tau_2^{3/2}}{\nu^{1/2}}\zeta(3) + \frac{2\pi^2}{3}T_2 + 4\pi\log\nu^{1/3} + I_D + \sum_{(p,q)=1}\mathcal{I}_{p,q} \tag{4.21}$$

This reproduces the results announced in Eqs (3.47,3.48) , up to the logarithmic term $4\pi\log\nu^{1/3}$ which is required for $SL(3,\mathbb{Z})$ invariance. We therefore conclude that the exact eight-dimensional thresholds can be written as:

$$\Delta_{tt} = \hat{E}_{\frac{3}{2}} - 2\pi\log(U_2|\eta(U)|^4) \quad , \quad \Delta_{\epsilon\epsilon} = \hat{E}_{\frac{3}{2}} + 2\pi\log(U_2|\eta(U)|^4) \tag{4.22}$$

5 Compactification of type IIB String Theory to $D < 8$

In this Section we shall investigate how the non-perturbative result of $D = 8$ can be extended to lower dimensions, focusing mainly on the seven-dimensional case. We will again propose a U-duality invariant expression for R^4 thresholds. We will show that it reproduces tree-level, D-instanton, fundamental string and (p,q)-string instanton contributions. The six-dimensional case will also be briefly discussed, where additional three-brane contributions are expected.

5.1 Perturbative compactification on a N-torus

Let G_{ij} be the σ-frame metric of the N-torus, and B_{ij}^{α} the associated antisymmetric tensors ($i = 1$ stands for the R-R antisymmetric tensor, $i = 2$ for the NS-NS one). We will separate the overall volume as

$$G_{ij} = V^{2/N}\tilde{G}_{ij} \quad , \quad \sqrt{\mathrm{det}G} = V \quad , \quad \mathrm{det}\tilde{G} = 1 \tag{5.1}$$

and define the $SL(2,\mathbb{Z})_\tau$ invariant scalar

$$\nu = \frac{1}{\tau_2 V^{4/N}} \tag{5.2}$$

The effective action of the IIB superstring toroidally compactified to $10-N$ dimensions is, in the Einstein frame,

$$S_{10-N} = \frac{1}{2k_{10-N}^2}\int d^{10-N}x\sqrt{-g_E}\left[R - \frac{N}{2(8-N)}\left(\frac{\partial\nu}{\nu}\right)^2 - \frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\tau_2^2} + \frac{1}{4}Tr(\partial\tilde{G}\partial\tilde{G}^{-1}) - \right. \tag{5.3}$$

$$-\frac{\nu}{4}\tilde{G}^{ik}\tilde{G}^{jl}\frac{(\partial B^1_{ij}+\tau\partial B^2_{ij})(\partial B^1_{kl}+\bar{\tau}\partial B^2_{kl})}{\tau_2}-\frac{\nu^2}{2.4!}\tilde{G}^{im}\tilde{G}^{jn}\tilde{G}^{kp}\tilde{G}^{lq}\partial C_{ijkl}\partial C_{mnpq}+\cdots\Bigg]$$

with $\kappa_{10-N}=\kappa_{10}/(2\pi)^{N/2}$. Here, we reinstated the four-form, which gives rise to moduli for $N\geq 4$.

We parametrize the $t_8t_8R^4$ threshold as:

$$S^{R^4}=\mathcal{N}_{10-N}\int d^{10-N}x\sqrt{-g_E}\left[\Delta_{tt}t_8t_8R^4\right]=\mathcal{N}_{10-N}\int d^{10-N}x\sqrt{-g_\sigma}\;\nu^{\frac{2N-4}{8-N}}V^{1-\frac{2}{N}}\left[\Delta_{tt}t_8t_8R^4\right] \tag{5.4}$$

where $\mathcal{N}_{10-N}=(2\pi)^N\mathcal{N}_{10}$. The one-loop perturbative correction can again be written in terms of the (N,N) torus lattice sum:

$$\nu^{\frac{2N-4}{8-N}}V^{1-\frac{2}{N}}\Delta^{1-loop}_{tt}=I_{N,N}=2\pi\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\tau_2^{\frac{N}{2}}\Gamma_{N,N}(G,B_N) \tag{5.5}$$

where

$$\tau_2^{\frac{N}{2}}\Gamma_{N,N}=\sqrt{G}\sum_{m^i,n^i\in Z}\exp\left[-\frac{\pi}{\tau_2}(G_{ij}+B_{N;ij})(m^i+n^i\tau)(m^j+n^j\bar{\tau})\right] \tag{5.6}$$

This integral has an infrared power divergence. It can be evaluated by the method of orbits [¶] [16]. Define the following sub-determinants, $d^{ij}=m^in^j-m^jn^i$. d^{ij} is an $N\times N$ antisymmetric matrix. The action of the Teichmüller group $SL(2,\mathbb{Z})$ decomposes into the following orbits:

- The trivial orbit, $m^i,n^i=0$, with a contribution

$$I^{tr}_{N,N}=2\pi\sqrt{G}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}1=\frac{2\pi^2}{3}V \tag{5.7}$$

- The degenerate orbits, with all d's being zero. In this case we can set $n^i=0$ and unfold the integration domain $\mathcal{F}$ onto the strip $\tau_1\in[-\frac{1}{2},\frac{1}{2}],\tau_2\in\mathbb{R}^+$. We obtain

$$I^d_{N,N}=2V\hat{\sum_{m^i}}\frac{1}{m^iG_{ij}m^j}=2\;V^{1-\frac{2}{N}}\hat{\sum_{m^i}}\frac{1}{m^i\tilde{G}_{ij}m^j} \tag{5.8}$$

Note that the sum is, up to a volume factor, the Eisenstein series $E_1(\tilde{G})$ for $SL(N,\mathbb{Z})$. It is indeed power-divergent for $N>2$ and in the Appendix we show (for the $N=3$ case) how the divergence can be subtracted. In the sequel we assume that this is carried out.

- The non-degenerate orbits, where at least one of the d^{ij} is non-zero. We can completely fix the modular $SL(2,\mathbb{Z})$ action

$$\begin{pmatrix}m^i\\n^i\end{pmatrix}\to\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}m^i\\n^i\end{pmatrix} \tag{5.9}$$

in order to unfold the integration domain to twice the upper-half plane. After Gaussian integration on τ, we obtain:

$$I^{n.d}_{N,N}=4\pi V^{1-\frac{2}{N}}\sum\frac{\exp\left[-2\pi V^{2/N}\sqrt{(m\cdot\tilde{G}\cdot m)(n\cdot\tilde{G}\cdot n)-(m\cdot\tilde{G}\cdot n)^2}+2\pi i(m\cdot B_N\cdot n)\right]}{\sqrt{(m\cdot\tilde{G}\cdot m)(n\cdot\tilde{G}\cdot n)-(m\cdot\tilde{G}\cdot n)^2}} \tag{5.10}$$

[¶] We would like to thank M. Henningson for collaborating in the calculation of this integral.

The summation is done over all sets of 2N integers, having at least one non-zero d^{ij}, modded out by the modular $SL(2,\mathbb{Z})$ action. This part is IR-finite.

5.2 Fundamental string world-sheet instantons on T^3

For $N=3$, we can "dualize" $B^{\alpha}_{ij}=\epsilon_{ijk}B^{\alpha k}$, where ϵ_{ijk} is the antisymmetric Levi-Civita symbol with $\epsilon_{123}=1$, and write the action in the simpler form:

$$S_7=\frac{1}{2k_7^2}\int d^7x\sqrt{-g_E}\left[R-\frac{3}{10}\left(\frac{\partial\nu}{\nu}\right)^2-\frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\tau_2^2}+\frac{1}{4}\mathrm{Tr}(\partial\tilde{G}\partial\tilde{G}^{-1})-\right. \tag{5.11}$$

$$\left.-\frac{\nu}{2}\tilde{G}_{ij}\frac{(\partial_\mu B^{1i}+\partial_\mu\tau B^{2i})(\partial_\mu B^{1j}+\bar{\tau}\partial_\mu B^{2j})}{\tau_2}+\cdots\right]$$

Again the scalar kinetic terms can be written in the form $\mathrm{Tr}(\partial M\partial M^{-1})/4$, where M is a symmetric 5×5 matrix with determinant 1, parametrizing the $SL(5,\mathbb{R})/SO(5)$ coset:

$$M=\nu^{3/5}\begin{pmatrix} g_{\alpha\beta} & g_{\alpha\beta}B^{\beta j}\\ g_{\alpha\beta}B^{\beta i} & \frac{\tilde{G}^{ij}}{\nu}+B^{\alpha i}B^{\beta j}g_{\alpha\beta}\end{pmatrix}\quad,\quad g_{\alpha\beta}=\frac{1}{\tau_2}\begin{pmatrix}1 & \tau_1\\ \tau_1 & |\tau|^2\end{pmatrix} \tag{5.12}$$

The U-duality group is $SL(5,\mathbb{Z})\supset O(3,3,\mathbb{Z})\times SL(2,\mathbb{Z})_\tau$, and acts as $M\to\Omega M\Omega^T$.

Using the dual representation of B_N we can also rewrite the contribution of the non-degenerate orbit to the one-loop threshold in a way that will be crucial for comparison with the full non-perturbative result. It amounts to trading the summation over the m,n integers modulo $SL(2,\mathbb{Z})$ for a summation over the $SL(2,\mathbb{Z})$ invariant integers d^{ij}, which can similarly be dualized as $d^{ij}=\epsilon^{ijk}d_k$. With these notations, one can show that

$$(m\cdot\tilde{G}\cdot m)(n\cdot\tilde{G}\cdot n)-(m\cdot\tilde{G}\cdot n)^2=d_i(\tilde{G}^{-1})^{ij}d_j\quad,\quad m\cdot B^N\cdot n=d_iB^i_N \tag{5.13}$$

We can then rewrite the contribution of the non-degenerate orbit as

$$I^{n.d}_{3,3}=4\pi V^{1/3}\overline{\sum}_{d_i}\frac{\exp\left[-2\pi V^{2/3}\sqrt{d\cdot\tilde{G}^{-1}\cdot d}+2\pi i d\cdot B^N\right]}{\sqrt{d\cdot\tilde{G}^{-1}\cdot d}} \tag{5.14}$$

This is the contribution of the fundamental $(1,0)$ string world-sheet instantons. There is a lot hidden in the summation sign, $\overline{\sum}_{d_i}$. We will make it more explicit presently, distinguishing the following three cases:

• Only one of the d_i is non-zero, say $d_1\neq0$, $d_2=d_3=0$. Then we can fix the $SL(2,\mathbb{Z})$ action by choosing the following representatives:

$$\begin{pmatrix} m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{pmatrix}=\begin{pmatrix}0 & k & j\\ 0 & 0 & p\end{pmatrix} \tag{5.15}$$

with $d_1=kp$, $p>0$, $0\le j<p$. The sum in Eq. (5.14) becomes

$$I^{n.d(1)}_{3,3}=4\pi V^{1/3}\sum_{\substack{k\neq0,p>0\\ j mod(p)}}\frac{\exp\left[-2\pi V^{2/3}|kp|\sqrt{\tilde{G}^{11}}+2\pi i kpB^1\right]}{|kp|\sqrt{\tilde{G}^{11}}} \tag{5.16}$$

$$= -8\pi \frac{V^{1/3}}{\sqrt{\tilde{G}^{11}}} \sum_{p=1}^{\infty} \text{Re} \log \left[1 - \exp\left(-2\pi V^{2/3} |k| \sqrt{\tilde{G}^{11}} + 2\pi i k B_N^1 \right) \right]$$

• Two out of the three d_i are non-zero, say d_1, d_2. Then we can choose the following representative:

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \begin{pmatrix} k_2 & k_1 & j \\ 0 & 0 & p \end{pmatrix} \tag{5.17}$$

with $d_1 = k_1 p$, $d_2 = k_2 p$, $p > 0$, $0 \le j < p$. The sum in Eq. (5.14) becomes in this case

$$I_{3,3}^{n.d(2)} = 4\pi V^{1/3} \sum_{\substack{k_i \neq 0, p>0 \\ j mod(p)}} \frac{\exp\left[-2\pi V^{2/3} p \sqrt{k_i \tilde{G}^{ij} k_j} + 2\pi i p k_i B_N^i \right]}{p \sqrt{k_i \tilde{G}^{ij} k_j}} \tag{5.18}$$

$$= 2\pi \sum_{k_i \neq 0} \frac{V^{1/3}}{\sqrt{k_i \tilde{G}^{ij} k_j}} \sum_{p=1}^{\infty} \exp\left(-2\pi V^{2/3} p \sqrt{k_i \tilde{G}^{ij} k_j} + 2\pi i p k_i B_N^i \right)$$

• Finally, consider the case where all the d_i are non-zero. Then we can choose the following representative

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 \\ 0 & n_2 & n_3 \end{pmatrix} \tag{5.19}$$

with $n_2 > 0$, $0 \le m_3 < n_3$. $d_1 = m_2 n_3 - m_3 n_2$, $d_2 = m_1 n_3$, $d_3 = m_1 n_2$. In this case we can show that a given (d_1, d_2, d_3) with greatest common divisor N corresponds to $\sum_{p|N} p$ equivalence classes of integers n, m. Thus we can write the contribution as

$$I_{3,3}^{n.d(3)} = 2\pi V^{1/3} \sum_{N=1}^{\infty} \sum_{p|N} \frac{p}{N} \sum_{(d_1,d_2,d_3)=1} \frac{\exp\left[-2\pi V^{2/3} N \sqrt{d \cdot \tilde{G}^{-1} \cdot d} + 2\pi i N d \cdot B \right]}{\sqrt{d \cdot \tilde{G}^{-1} \cdot d}} \tag{5.20}$$

$$= -2\pi \sum_{(d_1,d_2,d_3)=1} \frac{V^{1/3}}{\sqrt{d \cdot \tilde{G}^{-1} \cdot d}} \sum_{p=1}^{\infty} \log \left[1 - \exp\left(-2\pi V^{2/3} p \sqrt{d \cdot \tilde{G}^{-1} \cdot d} + 2\pi i p d \cdot B \right) \right]$$

5.3 Exact gravitational thresholds in $D = 7$ and $D = 6$

In addition to the perturbative contributions of the fundamental type IIB string, we expect instanton contributions from D-instantons and (p, q)-strings wrapped on the 3-torus. As in eight dimensions we can write down the result since the D-instanton contribution is known from the ten-dimensional result, and the (p, q) instanton contributions can be obtained from the contribution of the fundamental $(1, 0)$ string in Eq. (5.14). Thus we expect that

$$\Delta_{tt}^7 = \nu^{-9/10} f_{10}(\tau, \bar{\tau}) + 2\nu^{-2/5} E_{1,\text{reg}}^{SL(3)}(\tilde{G}) + \sum_{(p,q)=1} \mathcal{I}_{p,q}^7 \tag{5.21}$$

where

$$\mathcal{I}_{p,q}^7 = 2\pi \nu^{-2/5} \overline{\sum}_{d_i} \frac{\exp\left[-2\pi |p + q\tau| \frac{\sqrt{d \cdot \tilde{G}^{-1} \cdot d}}{\sqrt{\tau_2 \nu}} + 2\pi i d \cdot (q B_R - p B_N) \right]}{\sqrt{d \cdot \tilde{G}^{-1} \cdot d}} \tag{5.22}$$

We now show that, unsurprisingly, the full non-perturbative threshold (5.21) can be written in terms of the order-3/2 Eisenstein series for $SL(5,\mathbb{Z})$. This series is defined in terms of the $SL(5,\mathbb{R})/SO(5)$ matrix M in Eq. (5.12) as:

$$E^{SL(5)}_{\frac{3}{2}}(M) = 2\pi \int_0^\infty \left(\frac{1}{t^{5/2}} \hat{\sum_{m^i}} \exp\left[-\frac{\pi}{t} m^i M_{ij} m^j \right] - 1 \right) dt \tag{5.23}$$

$$= 2\pi \int_0^\infty \left(\frac{1}{t^{5/2}} \hat{\sum_{m^i,n_\alpha}} \exp\left[-\frac{\pi \nu^{3/5}}{t} \left(\frac{|m_1 + \tau m_2 + (B_R + \tau B_N)\cdot n|^2}{\tau_2} + \frac{n\cdot \tilde{G}^{-1}\cdot n}{\nu} \right) \right] - 1 \right) dt$$

Using the integral representation, and going through the same steps as in the $SL(3)$ case, we can establish that it is equal (up to an additive constant) to

$$E^{SL(5)}_{\frac{3}{2}} = \nu^{-9/10} f_{10}(\tau,\bar\tau) + 2\nu^{-2/5} E^{SL(3)}_{1,\text{reg}}(\tilde{G}) + \sum_{(p,q)=1} \hat{\mathcal{I}}^7_{p,q} \tag{5.24}$$

where

$$\hat{\mathcal{I}}^7_{p,q} = 2\pi \nu^{-2/5} \sum_{l=1}^{\infty} \hat{\sum}_{n^i} \frac{\exp\left[-2\pi l |p+q\tau| \frac{\sqrt{n\cdot\tilde{G}^{-1}\cdot n}}{\sqrt{\tau_2 \nu}} + 2\pi i l n\cdot (qB_R - qB_N) \right]}{\sqrt{n\cdot \tilde{G}^{-1}\cdot n}} \tag{5.25}$$

Separating the three cases, corresponding to one, two or three non-zero n^i, and taking out the greatest common divisor in the last case, we observe that

$$\hat{\mathcal{I}}^7_{p,q} = \mathcal{I}^7_{p,q} \tag{5.26}$$

and Eq. (5.24) coincides with Eq. (5.21), which proves our claim. This concludes our discussion of the seven-dimensional case.

We now briefly turn to the six-dimensional case. There we expect D3-instanton corrections in addition to the ones existing in higher dimensions. The scalar manifold is now $SO(5,5,\mathbb{R})/(SO(5)\times SO(5))$ and the U-duality group $SO(5,5,\mathbb{Z})$. It is more convenient to parametrize this manifold in terms of type IIA variables. Indeed, type IIA string compactified on T^4 can be viewed as the eleven-dimensional M-theory compactified on a five-torus. The scalars are the five-dimensional metric G_{ij} and the internal components of the three-form C_{ijk}, which can be dualized into a two index antisymmetric tensor $C_{ijk} = \epsilon_{ijklm}\tilde{C}^{lm}$. We can now construct the standard 10×10 symmetric $SO(5,5)$ matrix

$$M_6 = \begin{pmatrix} \det G\, G^{-1} - \tilde{C}G\tilde{C} & \tilde{G}\tilde{C} \\ -\tilde{C}G & G \end{pmatrix} , \tag{5.27}$$

in units of the 11D Planck scale. We again conjecture that the threshold should be given by the order 3/2 Eisenstein series, appropriately regularized:

$$\frac{\Delta^6_{tt}}{2\pi} = \int_0^\infty \left(\frac{1}{t^{5/2}} \hat{\sum_{m^i,n^i}} \exp\left[-\frac{\pi}{t} \left((m^i + \tilde{C}^{ik} n_k) G_{ij} (m^j + \tilde{C}^{jl} n_l) + (\det G)\, n_i G^{ij} n_j \right) \right] - t^{5/2} \right) dt \tag{5.28}$$

It remains to be shown that this expression reproduces the contribution of the ten-dimensional instantons, the D(p,q)-strings (that we can obtain from a one-loop $(1,0)$ string calculation), plus an extra piece that will be attributed to D3-brane instantons. We leave the further analysis of the six-dimensional case to a future publication.

6 Conclusions

In this paper we analysed in detail the threshold corrections of various R^4 terms in type IIB string theory compactified to eight and seven dimensions. The R^4 terms are BPS-saturated and receive perturbative contributions from one loop only, and from short N=8 multiplets. In ten dimensions, in addition to the perturbative contributions there are D-instanton corrections. In eight dimensions, (p,q)-string instantons can also contribute. We have calculated their contribution and shown that the full result is a order-3/2 Eisenstein series for $SL(3,\mathbb{Z})$, plus a order-one Eisenstein series for $SL(2,\mathbb{Z})$, invariant under the $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$ U-duality group in eight dimensions. This is in agreement with the M-theory calculation of [4]. Furthermore, it was noticed that $SL(3,\mathbb{Z})$ invariance requires the presence of a logarithmic term, absent in perturbation theory or in the instanton calculations. Its presence is tied to the logarithmic infrared divergence of the threshold in eight dimensions.

In seven dimensions the U-duality group is $SL(5,\mathbb{Z})$. The same type of instantons contribute as in the eight-dimensional case. We have calculated the exact threshold in this case and shown that it is given by a order-3/2 series for $SL(5,\mathbb{Z})$.

A more interesting case is the six-dimensional one, where D3-instantons are expected to contribute. We do expect again that the threshold will be given by a order-3/2 form ofor $SO(5,5,\mathbb{Z})$. This conjecture is also valid in lower dimensions, where one has to consider the order-3/2 Eisenstein series for the exceptional $E_{(n,n)}(\mathbb{Z})$ discrete groups. Checking this conjecture for $D\leq 6$ promises interesting insight into instanton calculus in string theory.

We make a final comment on the singularity structure of non-perturbative thresholds. Singularities in thresholds are due to states becoming massless. In our case, such singularities occur whenever a given term in the Eisenstein series diverges. This happens for specific values of the moduli. However, it is important to note that all these singularities can be mapped into the perturbative region $\tau_2\to\infty$. To put it otherwise, when the moduli are taken inside the fundamental domain of the U-duality group, then singularities occur at the boundaries. This is equivalent to stating that the U-duality is not broken by non-perturbative effects. This should be contrasted with cases where singularities appear inside the moduli space, and break the original duality group to a subgroup.

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Note added (1998). Shortly after this article was released, a proof of the conjectured ten-dimensional R^4 coupling Eq. 2.11 was given from heterotic-type II duality in four dimensions [21], and the perturbative non-renormalization theorem implied by this result was demonstrated from $N=2, D=8$ superspace techniques [22].

Note added (jan. 2010). The matrix M_6 in (5.27) and subsequent equation (5.28) have been corrected, in agreement with Eq. (28) of [23]. Moreover, in order to produce an eigenmode of the Laplacian, the summation in (5.28) should be restricted to null vectors with $n^i n_i=0$, as pointed out in [24].

Appendix A: Expansion and Regularization of the $SL(3)$ Eisenstein series

In this Appendix we give the expansion of the $SL(3)$ Eisenstein series E_s for arbitrary s. This is useful in order to derive the regularized form $\hat{E}_{\frac{3}{2}}$. The definition is

$$E_s = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \hat{\sum_{m^i \in Z}} \exp\left[-\frac{\pi}{t}\left(\sum_{i,j=1}^{3} m^i M^{ij} m^j\right)\right] \tag{A.1}$$

$$= \nu^{-s/3} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \hat{\sum_{m^i \in Z}} \exp\left[-\frac{\pi}{t}\left(\frac{m_3^2}{\nu} + \frac{|m_1 + m_2\tau + m_3 B|^2}{\tau_2}\right)\right]$$

Going through the same steps as in Section 4 we derive the following expansion:

$$E_s = 2\nu^{-s/3}\tau_2^s\zeta(2s) + 2\sqrt{\pi}T_2\left(\tau_2\nu^{1/3}\right)^{\frac{3}{2}-s}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1) + \tag{A.2}$$

$$+2\pi\nu^{\frac{2s}{3}-1}\frac{\zeta(2s-2)}{s-1} + I_D^s + \hat{\sum_{p,q\in Z}} I_{p,q}^s$$

where

$$I_D^s = 2\frac{\sqrt{\tau_2}}{\nu^{s/3}}\frac{\pi^s}{\Gamma(s)}\sum_{m,n\neq 0}\left|\frac{m}{n}\right|^{s-\frac{1}{2}} e^{2\pi i m n \tau_1} K_{s-\frac{1}{2}}(2\pi|mn|\tau_2) \tag{A.3}$$

$$I_{p,q}^s = 2\frac{\nu^{(s-3)/6}}{\tau_2^{(s-1)/2}}\frac{\pi^s}{\Gamma(s)}\sum_{m\neq 0}\left|\frac{p+q\tau}{m}\right|^{s-1} e^{2\pi i m(qB_1-(p+q\tau_1)B_2/\tau_2)} K_{s-1}\left(2\pi|m|\frac{|p+q\tau|}{\sqrt{\nu\tau_2}}\right) \tag{A.4}$$

where the K Bessel function arises through its integral representation:

$$\int_0^\infty \frac{dx}{x^{1-\lambda}} e^{-b/x-cx} = 2\left|\frac{b}{c}\right|^{\lambda/2} K_\lambda(2\sqrt{|bc|})\ , \quad K_\lambda(x) = K_{-\lambda}(x)\ , \quad K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x} \tag{A.5}$$

As a function of s, E_s has potential simple poles at $s = 1/2, 1, 3/2$. Indeed, $\zeta(s)$ has a simple pole at $s = 1$ and $\Gamma(s)$ a simple pole at $s = 0$:

$$\zeta(1+\epsilon) = \frac{1}{\epsilon} + \gamma + \mathcal{O}(\epsilon)\ , \quad \Gamma(\epsilon) = 1/\epsilon - \gamma + \mathcal{O}(\epsilon) \tag{A.6}$$

where $\gamma = 0.577215...$ is the Euler constant. From this we find that the residues $\mathcal{R}$ of the simple poles of E_s are

$$\mathcal{R}_{1/2} = \mathcal{R}_1 = 0 \quad , \quad \mathcal{R}_{3/2} = 2\pi \tag{A.7}$$

so that E_1 and $E_{1/2}$ are actually well defined. On the other hand, for $s = 3/2$, using Eq. (4.4) we obtain

$$\hat{E}_{\frac{3}{2}} = 2\zeta(3)\frac{\tau_2^{3/2}}{\nu^{1/2}} + \frac{2\pi^2}{3}T_2 + 4\pi\log\nu^{1/3} + I_D^{3/2} + \hat{\sum_{p,q\in Z}} I_{p,q}^{3/2} \tag{A.8}$$

with

$$I^{3/2}_{p,q} = 4 \sum_{m\neq 0} \frac{1}{|m|} \exp\left[-2\pi|m|T_2|p+q\tau| + 2\pi i m(qB_1 - (p+q\tau_1)B_2/\tau_2)\right] \tag{A.9}$$

Finally we can rewrite the above result using the $SL(2,\mathbb{Z})$ invariant form $f_{10}(\tau,\bar\tau)$ introduced in Eq. (2.12) as

$$\hat{E}_{\frac{3}{2}} = \frac{f_{10}(\tau,\bar\tau)}{\nu^{1/2}} + 2\pi \log \nu^{1/3} + \hat{\sum_{p,q\in Z}} I^{3/2}_{p,q} \tag{A.10}$$

Since we are also interested in other divergent $SL(3,\mathbb{Z})$ Eisenstein series, we would like to regularize the sum in a generic way, namely by introducing a dimensionful parameter μ. This can be done by inserting a regulating function in Eq. (A.1):

$$E^{\mu}_s = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \hat{\sum_{m^i\in Z}} \exp\left[-\frac{\pi}{t}\left(\sum_{i,j=1}^{3} m^i M^{ij} m^j\right)\right] R_\mu(t) \tag{A.11}$$

$$= \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} \hat{\sum_{m^i\in Z}} \exp\left[-\frac{\pi\nu^{1/3}}{t}\left(\frac{m_3^2}{\nu} + \frac{|m_1+m_2\tau+m_3B|^2}{\tau_2}\right)\right] R_\mu(t)$$

Going through the same steps of the calculation we obtain

$$E_{s,\mathrm{reg}} = 2\frac{\pi^s}{\Gamma(s)}\left[\int_0^\infty dt\ t^{s-1} R_\mu(1/t) \sum_{m=1}^\infty e^{-\pi t\nu^{1/3} m^2/\tau_2} + \frac{\sqrt{\tau_2}}{\nu^{1/6}} \int_0^\infty dt\ t^{s-3/2} R_\mu(1/t) \sum_{m=1}^\infty e^{-\pi t \nu^{1/3}\tau_2 m^2}\right.$$

$$\left. + \nu^{-1/3} \int_0^\infty dt\ t^{s-2} R_\mu(1/t) \sum_{m=1}^\infty e^{-\pi t \nu^{-2/3} m^2}\right] + I^s_D + \hat{\sum_{p,q\in Z}} I^s_{p,q} + \cdots$$

where the dots stand for setting the regulating function to 1 inside the finite contributions I^s_D and $I^s_{p,q}$. For s=3/2 we can choose

$$R_\mu = 1 - e^{-\frac{\pi}{\mu^2 t}} \tag{A.12}$$

to obtain

$$E^{\mu}_{\frac{3}{2}} = -\frac{1}{2}\log\mu^2 + \gamma - \log 2 + \frac{1}{3}\log\nu + \frac{f_{10}(\tau,\bar\tau)}{\nu^{1/2}} + \hat{\sum_{p,q\in Z}} I^{3/2}_{p,q} + \mathcal{O}(\mu^2) \tag{A.13}$$

where we have used

$$\sum_{m=1}^{\infty}\left[\frac{1}{m} - \frac{1}{\sqrt{m^2+\kappa^2}}\right] = \gamma + \log(\kappa/2) + \mathcal{O}(1/\kappa) \tag{A.14}$$

Finally, a third way of regularizing the series, which is useful for power-divergent series, is to subtract "by hand" the divergent piece. In particular, we can define the order-1 Eisenstein series as

$$E_{1,\mathrm{reg}} = \pi \int_0^\infty \left(\frac{1}{t^2} \hat{\sum_{m^i\in Z}} \exp\left[-\frac{\pi}{t}\left(\sum_{i,j=1}^{3} m^i M^{ij} m^j\right)\right] - \frac{1}{\sqrt{t}}\right) dt\ . \tag{A.15}$$

This way of regulating is motivated from perturbative threshold corrections, and manifestly preserves the $SL(3, \mathbb{Z})$ symmetry.

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